

CHANGEVR,  
A REDUCE Facility  
to  
Perform Change of Independent Variable(s)  
in  
Differential Equations

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# 1 Introduction

The mathematics behind the change of independent variable(s) in differential equations is quite straightforward. It is basically the application of the chain rule. If the dependent variable of the differential equation is  $F$ , the independent variables are  $x_i$  and the new independent variables are  $u_i$  (where  $i=1\dots n$ ) then the first derivatives are:

$$\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial u_j} \frac{\partial u_j}{\partial x_i}$$

We assumed Einstein's summation convention. Here the problem is to calculate the  $\partial u_j / \partial x_i$  terms if the change of variables is given by

$$x_i = f_i(u_1, \dots, u_n)$$

The first thought might be solving the above given equations for  $u_j$  and then differentiating them with respect to  $x_i$ , then again making use of the equations above, substituting new variables for the old ones in the calculated derivatives. This is not always a preferable way to proceed. Mainly because the functions  $f_i$  may not always be easily invertible. Another approach that makes use of the Jacobian is better. Consider the above given equations which relate the old variables to the new ones. Let us differentiate them:

$$\begin{aligned} \frac{\partial x_j}{\partial x_i} &= \frac{\partial f_j}{\partial x_i} \\ \delta_{ij} &= \frac{\partial f_j}{\partial u_k} \frac{\partial u_k}{\partial x_i} \end{aligned}$$

The first derivative is nothing but the  $(j, k)$  th entry of the Jacobian matrix.

So if we speak in matrix language

$$\mathbf{1} = \mathbf{J} \cdot \mathbf{D}$$

where we defined the Jacobian

$$\mathbf{J}_{ij} \triangleq \frac{\partial f_i}{\partial u_j}$$

and the matrix of the derivatives we wanted to obtain as

$$\mathbf{D}_{ij} \triangleq \frac{\partial u_i}{\partial x_j}$$

If the Jacobian has a non-vanishing determinant then it is invertible and we are able to write from the matrix equation above:

$$\mathbf{D} = \mathbf{J}^{-1}$$

so finally we have what we want

$$\frac{\partial u_i}{\partial x_j} = [\mathbf{J}^{-1}]_{ij}$$

The higher derivatives are obtained by the successive application of the chain rule and using the definitions of the old variables in terms of the new ones. It

can be easily verified that the only derivatives that are needed to be calculated are the first order ones which are obtained above.

## 2 How to Use CHANGEVR

**This facility requires the matrix package to be present in the session.** So if it is not autoloaded in your REDUCE implementation, say

```
LOAD_PACKAGE MATRIX;
```

in the REDUCE environment. Then load CHANGEVR by the statement:

```
LOAD_PACKAGE CHANGEVR$
```

Now the REDUCE function CHANGEVAR is ready to use. **Note: The package is named CHANGEVR, but the function has the name CHANGEVAR.** The function CHANGEVAR has (at least) four different arguments. Here we give a list them:

- **FIRST ARGUMENT**

Is a list of the dependent variables of the differential equation. They shall be enclosed in a pair of curly braces and separated by commas. If there is only one dependent variable there is no need for the curly braces.

- **SECOND ARGUMENT**

Is a list of the **new** independent variables. Similar to what is said for the first argument, these shall also be separated by commas, enclosed in curly braces and the curly braces can be omitted if there is only one new variable.

- **THIRD ARGUMENT**

Is a list of equations separated by commas, where each of the equation is of the form

$$\textit{old variable} = \textit{a function in new variables}$$

The left hand side cannot be a non-kernel structure. In this argument the functions which give the old variables in terms of the new ones are introduced. It is possible to omit totally the curly braces which enclose the list. **Please note that only for this argument it is allowed to omit the curly braces even if the list has more than one items.**

- **LAST ARGUMENT**

Is a list of algebraic expressions which evaluates to differential equations, separated by commas, enclosed in curly braces. So, variables in which differential equations are already stored may be used freely. Again it is possible to omit the curly braces if there is only **one** differential equation.

If the last argument is a list then the result of `CHANGEVAR` is also a list.

It is possible to display the entries of the inverse Jacobian, explained in the introduction. To do so, turn `ON` the flag `DISPJACOBIAN` by a statement:

```
ON DISPJACOBIAN;
```

### 3 AN EXAMPLE.....The 2-dim. Laplace Equation

The 2-dimensional Laplace equation in cartesian coordinates is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now assume we want to obtain the polar coordinate form of Laplace equation. The change of variables is:

$$x = r \cos \theta, \quad y = r \sin \theta$$

The solution using `CHANGEVAR` (of course after it is properly loaded) is as follows

```
CHANGEVAR({u},{r,theta},{x=r*cos theta,y=r*sin theta},  
          {df(u(x,y),x,2)+df(u(x,y),y,2)} )
```

Here we could omit the curly braces in the first and last arguments (because those lists have only one member) and the curly braces in the third argument (because they are optional), but you cannot leave off the curly braces in the second argument. So one could equivalently write

```
CHANGEVAR(u, {r, theta}, x=r*cos theta, y=r*sin theta,
           df(u(x,y), x, 2)+df(u(x,y), y, 2) )
```

If you have tried out the above example, you will notice that the denominator contains a  $\cos^2 \theta + \sin^2 \theta$  which is actually equal to 1. This has of course nothing to do with the `CHANGEVAR` facility introduced here. One has to overcome these pattern matching problems by the conventional methods `REDUCE` provides (a `LET` statement, for example, will fix it).

Secondly you will notice that your `u(x,y)` operator has changed to `u(r,theta)` in the result. Nothing magical about this. That is just what we do with pencil and paper. `u(r,theta)` represents the transformed dependent variable.

## 4 ANOTHER EXAMPLE. . . . . An Euler Equation

Consider a differential equation which is of Euler type, for instance:

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$$

Where prime denotes differentiation with respect to  $x$ . As is well known, Euler type of equations are solved by a change of variable:

$$x = e^u$$

So our `CHANGEVAR` call reads as follows:

```
CHANGEVAR(y, u, x=e**u, x**3*df(y(x), x, 3)-
           3*x**2*df(y(x), x, 2)+6*x*df(y(x), x)-6*y(x))
```