COMPACT: Reduction of a Polynomial in the Presence of Side Relations

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COMPACT is a package of functions for the reduction of a polynomial in the presence of side relations. The package defines one operator COMPACT whose syntax is:

COMPACT(*<expression>*, *<list>*):*<expression>*

<expression> can be any well-formed algebraic expression, and *<list>* an expression whose value is a list of either expressions or equations. For example

```
compact(x**2+y**3*x-5y,{x+y-z,x-y-z1});
compact(sin(x)**10*cos(x)**3+sin(x)**8*cos(x)**5,
        {cos(x)**2+sin(x)**2=1});
let y = {cos(x)**2+sin(x)**2-1};
compact(sin(x)**10*cos(x)**3+sin(x)**8*cos(x)**5,y);
```

COMPACT applies the relations to the expression so that an equivalent expression results with as few terms as possible. The method used is briefly as follows:

1. Side relations are applied separately to numerator and denominator, so that the problem is reduced to the reduction of a polynomial with respect to a set of polynomial side relations.

2. Reduction is performed sequentially, so that the problem is reduced further to the reduction of a polynomial with respect to a single polynomial relation.

3. The polynomial being reduced is reordered so that the variables (ker-

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nels) occurring in the side relation have least precedence.

4. Each coefficient of the remaining kernels (which now only contain the kernels in the side relation) is reduced with respect to that side relation.

5. A polynomial quotient/remainder calculation is performed on the coefficient. The remainder is used instead of the original if it has fewer terms.

6. The remaining expression is reduced with respect to the side relation using a "nearest neighbor" approach.

As with the traveling salesman problem, a nearest neighbor approach to reduction does not necessarily achieve an optimal result. In most cases it will be within a factor of two from the optimal result, but in extreme cases it may be much further away.

Another source of sub-optimal results is that the given expression is reduced sequentially with respect to the side relations. So for example in the case

```
compact((a+b+c)*(a-b-c)*(-a+b-c)*(-a-b+c),
        {x1=a+b+c,x2=a-b-c,x3=-a+b-c,x4=-a-b+c})
```

the expression is actually $x_1x_2x_3x_4$, but any given relation cannot reduce the size of the expanded form $a^4 - 2a^2b^2 - 2a^2c^2 + b^4 - 2b^2c^2 + c^4$ of the original expression, and so the final result is far from optimal.

The only other program we have heard about that considers the compaction problem is that of Hornfeldt [**?**]. However, Hornfeldt reorders expressions so that the kernels in a side relation have highest order. Consequently, their coefficients are polynomials rather than integers or other constants as in our approach. Furthermore, it is not clear just how general Hornfeldt's approach is from his description, since he only talks about sine and cosine substitutions.

There are a number of projects that this work immediately suggests. For example:

1. How does one do the reduction with the side relations in parallel? The above example shows this is necessary for an optimal solution.

2. Should one reduce the side relations to a Groebner or other basis before doing any reduction?

3. Should one check for the consistency of the basis?

4. How does one do factorization and gcds on a polynomial whose variables are related by a set of side relations?

The author would be interested in hearing from anyone wishing to work with him on any of these problems.

References